AN INVENTORY MODEL FOR TWO ITEMS WHICH ARE PARTIALLY INTERCHANGEABLE

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AN INVENTORY MODEL FOR TWO ITEMS WHICH

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NOMENCLATURE

The following is a list of letters and symbols that are used throughout this thesis:

A	Fixed cost incurred each time an order is placed.
B	Average/expected number of backorders at any point in time
C	Unit cost of an item.
C_{ij}	Unit conversion cost from product i to product j.
C_{sj}	Transportation cost from source to inventory j.
D	Rate of demand.
E	Expected number of backorders incurred in a year.
F	Expected on-hand plus on-order inventory.
H	Expected allocation costs.
I	Carrying charge.
J	Transportation cost per unit from factory to warehouse.
K	Cost function for system with free and instantaneous redistribution and assembly of product 1.
L	Lagrangian function.
M	Inventory level at which redistribution is considered.
P	Rate of production; complementary cumulative density function prefix.
P_{out}	Probability that the system is out of stock at any point in time.

Q Reorder quantity.

R Expected cost associated with hitting an M level (when double subscripted); function notation prefix for expected redistribution costs.

S Safety stock.

T Length in time of one period.

TC Total cost function for system.

T_{cj} Cutoff time for inventory j.

T_{mj} Time between receipt of first increment of an allocation to achievement of maximum inventory level for the period.

T_O Time of arrival of first increment of previous allocation.

U Function notation prefix used in redistribution rule development.

V Average annual transportation cost for the deterministic system; function notation prefix used in redistribution rule development.

W Total amount of warehouse space available; expected shortage cost when subscripted and used as function notation prefix.

X General purpose variable.

Y General prupose variable; function notation prefix used in development of allocation rule.

Z Expected annual costs of redistribution and stockouts.

a Cost per unit of assembling components into finished products.

b Number of backorders incurred during a period.

c Subscript.

d Differential prefix.

e Mathematical quantity.

f Function notation prefix.

g Maximum inventory level achieved during a period (when subscripted).

h Function notation prefix.

i Subscript or general purpose variable.

j Subscript or general purpose variable.

k System reorder point.

l General purpose variable.

m Subscript.

p Probability density function prefix.

q Allocation quantities for deterministic model (when subscripted).

r Number of units demanded in the system since the last allocation.

s Number of backorders at time of arrival of first increment of a new order.

t Instantaneous measure of time; time since previous allocation when an inventory reaches its M level.

t_{ij} Time for which backorder j exists in period i.

t_a Time measured from when an M level is reached until arrival of the first increment of the next allocation.

t_i Time, measured from the time inventory j reaches M_j, until cutoff time, T_{ci}.

u General purpose variable; on-hand inventory at cutoff time.

v Transportation cost per cycle for deterministic model.

w Amount of warehouse space consumed per unit (when subscripted); demand against an inventory during redistribution lead time.

x Redistribution quantity (when double subscripted); general purpose variable.

y On-hand inventory at any point in time; on-hand inventory when an inventory reaches its M level; on-hand inventory when an allocation is to be made.

z Amount demanded in time T_{mj}.

Greek Letters

α Critical probability of incurring one or more backorders in redistribution lead time (when subscripted); function notation prefix.

π Fixed cost associated with each backorder.

$\hat{\pi}$ Cost proportional to length of time for which backorder exists.

ζ Time since observation of the inventory system was begun.

ξ Total number of backorders incurred during a period.

Δ Unit years of shortage incurred during a period.

τ Production lead time; redistribution lead time (when double subscripted).

Γ Function notation prefix associated with redistribution rule development.

θ On-hand inventory immediately after arrival of a redistribution.

ψ Part of expected cost of redistribution.

ρ Lagrange multiplier.

SUMMARY

The objective of this thesis is to develop a model of a single-source, two-product inventory system where both products are stored in the same factory warehouse and are partially interchangeable through minor modifications. Although the system consists of two products and one warehouse, the approach used in formulating the model is one in which one source supplies two separate warehouses with the same product. This approach is possible because the redistribution between the two products within the same warehouse is analogous to redistribution of one product between two separate warehouses since the time and cost factors are similar.

The procedure for the development of the model is to first develop a model for the system when demands against the inventories are deterministic. Following that, a model is developed for the system when the demands against the inventories are stochastic in nature. A general approach to optimization of each model is discussed.

The development of the deterministic model consists of the formulation of a total cost function for the system and the development of a constraint representing the warehouse space restriction. It is shown that redistribution for the system is never an optimal policy when demands are deterministic and is therefore excluded from the model.

The development of the stochastic model is conducted in three phases: (1) the formulation of a total cost function and a warehouse

space constraint, (2) the formulation of a function to determine the redistribution rule, and (3) the formulation of a function to determine the amount of each order to allocate to each inventory. To facilitate an economical solution to the model, the total cost function is then reduced in size and complexity by making some simplifying assumptions and approximations.

Certain conclusions can be made as a result of this study. It is demonstrated that the multi-warehouse approach to formulating a model is appropriate for use in single warehouse, multi-product systems where products are interchangeable. In addition, the models developed are of sufficient complexity to make the optimization process difficult. Although the Lagrange Mulitplier method can be used for the deterministic model, the nonlinear simultaneous equations resulting from this method must be solved using the digital computer. As an alternative, computerized search techniques may be employed to find a near-optimal policy. Search techniques must also be employed to find a near-optimal policy for the stochastic model. Because the stochastic model is composed of three separate cost functions, the optimization process is made simpler.

CHAPTER I

INTRODUCTION

In recent years inventory theory has evolved as a discipline of considerable importance. Since World War II the pressure for operating capital has made business increasingly aware of inventory as a form of investment. Obviously, capital that is invested in inventory is not available for investment in other activities. On the other hand, some capital must be invested in inventory in order to meet customer demand. If there is insufficient inventory to fill customer demands, then there are costs to the manufacturer in terms of lost sales, expedited production, redistribution, and other forms. On the other hand, the greater the amount of inventory, the greater the costs to the manufacturer in terms of taxes, storage requirements, lost investment opportunities elsewhere, and others. In addition, if the product happens to be something that becomes obsolete or unsalable with time, then too much inventory can result in costs to the manufacturer in terms of unsalable products. The basic problem of inventory policy, then, is to strike a balance between operating savings and the cost and capital requirements associated with inventory stockage levels.

Need for Inventory Control

The question might be posed, why has the need for sound methods of inventory management increased so significantly in the past 25 years? Magee and Boodman (5) have indicated that in the past businessmen have been able to use an intuitive understanding of their business needs to achieve a reasonably balanced inventory policy. However, as businesses grew in complexity, intuition became less adequate as a means of maintaining an economical balance. Business executives became more specialized in their jobs and became farther removed from direct operations. In addition, technology became more advanced; i.e., the tools for operating complex inventory systems are much more advanced now than they were 25 years ago. These include advanced clerical procedures, better communications and transportation facilities, and high speed digital computers capable of solving heretofore unsolvable inventory models.

Other factors have also lead to the rapid development of scientific methods of inventory control. In addition to organizations being larger and more complex, they, for the most part, operate on a much more narrow margin of profit than before. Because of this, business management can no longer afford the luxury of "seat-of-the-pants" intuitive decision making. Although it cannot be said that scientific inventory control eliminates all risks in decision making, it can be said that it reduces the risks by enabling the manager to make a more knowledgeable decision. Associated with the margin of profit factor are the pressure for capital and growth of return on investment as a measure of business performance (5). Thus capital budgeting methods have caused an increased consciousness of the amount invested in inventories. Also, the trend toward heavy fixed investment to reduce direct-labor cost versus pressure from labor for employment stability have forced more careful future planning of inventories. Magee and

Boodman describe in detail other implications of inventory management that provide background for the development of scientific production and inventory control.

Classification of Inventory Models

The basic vehicle for analysis of inventory systems is a mathematical model. Because there are so many different types and sizes of inventory systems, there are numerous ways of describing them mathematically. These models vary from simple to very complex. In most cases certain approximations and simplifications have to be made in constructing the model since it is seldom feasible or economical to represent the real world with complete accuracy. In the final analysis, however, two fundamental questions must be answered by all models of inventory systems: (a) when to replenish the inventory, and (b) how much to order for replenishment.

The complete classification of an inventory system can include many descriptive factors. A discussion of these can be found in Chapter I of (11). As an example, one factor is the echelon structure of the system. The system is single echelon or multiechelon depending on whether an item is stocked at a single point or at many points in the system. In most cases, however, inventory models are classified in terms of the type demands (deterministic or stochastic) against the inventories, the type of reordering policy used (lot size or order-to-R), the echelon structure of the system, the type of review or reporting of demands against the system (periodic review or transactions reporting), and the type of stockouts permitted (backorders or lost sales). However, it takes much more than these factors to totally describe a system.

The inventory system that we will be concerned with in this study can be described as a single echelon, multi-warehouse system in which the demands against each inventory are stochastic in nature. Demands that occur when the system is out of stock are backordered, and redistribution between warehouses is permitted. The method of transactions reporting will be used, and a set lot size will be ordered each time the system inventory position (on-hand inventory plus on-order inventory minus backorders) reaches a certain reorder point. The system will be described in detail in the following section.

Description of the Problem

Our system consists of two related products that are stored in the same factory warehouse to meet future demands (see Figure 1). Both products are used essentially for the same type function and, in their finished form, are identical in all respects with the exception that one product receives additional customization to the purchaser's specifications and the other is plain. The plain product is referred to as product 1 and the customized product as product 2.

Each product is manufactured by the same factory and by essentially the same process, differing only in the final production stage. The manufacturing process produces an identical number of components each time the process is run. A portion of the components is then allocated to the product 1 inventory and the remainder allocated to the product 2 inventory. Those components allocated to the

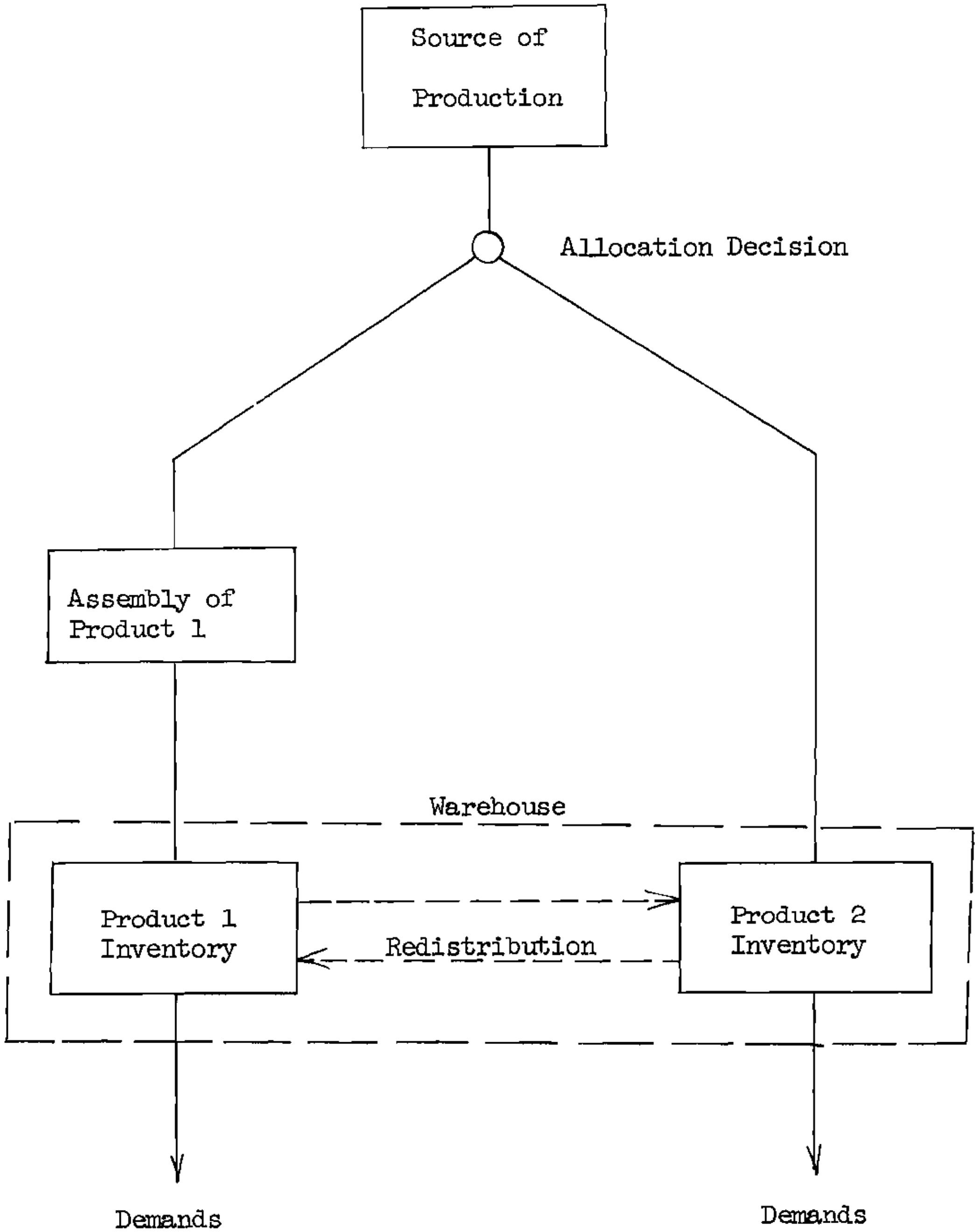

Figure 1. System Diagram

product 2 inventory must be stored in the warehouse unassembled. When a demand for product 2 is received, the items are then customized to the specifications and assembled. They must be unassembled for customizing. Those components allocated to the product 1 inventory are first assembled and then stored in the warehouse in their finished form. When a demand for product 1 is received, there is no customizing or assembly process to be undergone prior to filling the order.

Both inventories are stored in the same warehouse. There is only one warehouse available for storage of the two inventories, and its limited amount of floor-space imposes a constraint on the total amount of system inventory that may be stored. That is, the two types of products must compete with each other with respect to the limited warehouse space.

Normal production costs associated with the two types of products differ in that an additional assembly cost is involved with product 1 whereas the costs of customizing and assembly of product 2 are borne by the purchaser in the form of a higher market price. In other words, the manufacturer pays the assembly cost for product 1 while the purchaser pays both the assembly and the customizing costs for product 2.

For the initial formulation of the model the rate of demand for each product will be assumed to be known and constant with respect to time. A second formulation will be made for demands which are stochastic in nature but reasonably predictable in accordance with known probability distributions. For both deterministic demands and stochastic demands any demand against either inventory that cannot be met because of insufficient on-hand inventory will be backordered until there is sufficient inventory on hand to fill the order. Also, in both the deterministic-demand model and the stochastic-demand model redistribution between the two inventories will be considered as a possible alternative to incurring stockouts.

Because of the similarities between product 1 and product 2, redistribution between the two inventories can be effected, but at a cost in time and money. As an example, if it is anticipated that future demands for product 1 will exceed the on-hand inventory of product 1, then a portion of product 2 inventory can be converted to product 1 items by assembling the unassembled components. This process would require time and money, and would be at the expense of the manufacturer. Similarly, if shortages are anticipated for the product 2 inventory, a portion of the product 1 inventory can be converted to product 2 items by disassembling, customizing, and reassembling the proudct 1 items. This process would take more time and result in a much higher cost to the manufacturer, as disassembly is assumed to be more expensive than assembly. As mentioned earlier, the redistribution process is an alternative to incurring backorders, but the relative costs involved must be considered in the establishment of the optimal operating policy.

Since redistribution between the two inventories involves an expense to the manufacturer in terms of both time and money and since demands against the two inventories are independent of each other, the system can be thought of as consisting of two warehouses. Normally, in a multi-warehouse situation the redistribution costs involved are transportation costs and shipping time between warehouses. This problem has only one warehouse, and thus transportation costs and shipping time between the two inventories are not factors; however, the labor cost involved in converting from one product to another and the time involved in such a process can be substituted for the transportation cost and shipping time respectively. This problem can therefore be classified as having single-echelon, multi-warehouse characteristics with the supply of products to each inventory (warehouse) coming from a single source.

As is often the case with factory warehouses, orders to replenish the respective inventories are not received in total lots at one time at the warehouse but, instead, are sent to the warehouse in partial lots as they are produced at the factory. It will be assumed for this problem that the rate of production, P, will be finite, constant with respect to time, and independent of the quantity ordered. As an example, after a fixed production set-up time known as production lead time, it will begin to produce the products (in the form of separate components) at the rate of P units per year. As the products are produced, they are sent immediately to the respective inventories in the amounts determined by the allocation decision rule. Components destined for the product 1 inventory will have to undergo the additional stage of assembly before becoming a part of the inventory. After the entire order of Q units has been produced by the factory, the production process will be stopped until another order for Q units is received from the warehouse.

It will be assumed for this problem that orders will be made on a system-wide basis by the central control point which is kept constantly informed of the inventory levels through transactions reporting; that is, when a demand against one of the inventories is filled, the central control point is immediately notified of the depletion of that inventory by the amount of the filled demand. When the system reorder point, k, is reached, an order for Q units will be sent to the factory. The reorder point, k, will be based on the inventory position of the total inventory system, i.e., the combination of both the product 1 and product 2 inventories. Its value will be established as part of the optimal operating policy to be developed in Chapter IV.

Two alternative methods of system operation warrant consideration. One alternative is to completely eliminate any redistribution between the two inventories and operate the system as two separate inventory systems with each inventory having its own reorder point and reorder quantity. This approach might be justified when the redistribution costs are so high that redistribution would never be an optimal course of action. In such a situation the involved computation procedures for redistribution can be eliminated entirely and models developed for two separate inventories. A second alternative method of operation would be to treat the system as consisting of only one inventory where only unassembled components are stored in the warehouse. When a demand for product 1 occurs, the necessary components are assembled to fill the demand; and when a demand for product 2 occurs, the necessary components are customized and assembled. This method would also eliminate the need for redistribution computation and would be appropriate in a situation where transportation and assembly costs and product-to-customer lead time are sufficiently small to warrant its use in lieu of separate inventories for product 1 and product 2.

In view of the above two alternative methods of operation, two further assumptions must be made to justify the use of our method of operation. One assumption is that the costs of redistribution are in ranges which cause redistribution to frequently be an optimal course of action. This would make the redistribution computation necessary. The second assumption is that the relative transportation, assembly, and time costs involved when a demand for product 1 occurs are sufficiently high to warrant the maintenance of two separate inventories of assembled and unassembled components.

Applications of the Model

This section will discuss two potential applications of the model. Consider a production process for plastic boxes that are used primarily as containers for hardware or jewelry. The system consists of two types of plastic boxes that are stored in the same factory warehouse to meet future demands. Both boxes are the same with the exception that one box is engraved on the top to the buyer's specifications and the other is sold plain. The engraved box will be referred to as the custom model and the plain box as the standard model. In order for engraving to take place for the custom model, the box must be in an unassembled state. Therefore, the component tops and bottoms for the custom model are stored in the warehouse unassembled and the standard boxes are stored assembled. If the standard inventory is out of stock when a demand against it occurs, it can be seen that redistribution can be effected by assembling some of the tops and bottoms of the custom inventory to form standard models. Redistribution can also be effected from standard to custom inventories, but this would entail disassembling standard models. Experience has demonstrated a high breakage rate in the disassembly process. Therefore, redistribution from the standard to custom inventories would be at a much higher cost and probably would not take place. The model treated in this investigation adequately describes the system if we can let product 1 represent the standard model and product 2 represent the custom model.

As a second example, consider a firm that stocks steel beams to meet anticipated future needs. Suppose a particular cross-section of beam is normally stocked in two standard lengths, s (short) and l (long). The same production process produces the beams with the exception that the ends of the short beams are finished and the ends of the long beams are left unfinished. The short beams are sold in their finished form without any further modification. The long beams are cut in length and finished to the buyer's specifications. If a demand for s occurs and the s inventory is out of stock, redistribution can be effected from the l inventory to the s inventory by shortening l and finishing the ends. If l is out of stock when a demand occurs, redistribution can be effected from s to l only if the buyer wants a beam that is less than or equal to the length of s. If the buyer wants a longer beam than s, redistribution cannot be effected. The model treated in this investigation describes this system if we let product 1 represent short beams and product 2 represent long beams. The only difference lies in the redistribution from s to l if the buyer wants a beam longer than s. This can be rectified by a slight modification in model development.

Method of Optimization

Having formulated a model representing the system, the next step is to use it as an aid in developing a suitable operating doctrine for the system. In most cases it is desirable to arrive at a policy that will either maximize profits or minimize costs. Ideally, it would be desirable to use analytical techniques to arrive at a doctrine. For the simpler models this might involve merely the taking of partial derivatives, setting them equal to zero, and solving for the desired quantities. Other more complex models might be optimized through dynamic programming. Still other models might be of such complexity, even after reasonable simplifying assumptions are made, that the only recourse is to use computer simulation techniques to examine a sample number of doctrines. This is the least desirable method since it is seldom possible to determine the optimal doctrine. The most that can be expected from simulation is to select a best policy from the ones that are examined.

For our problem the objective will be to minimize a total cost function in order to determine the optimal values for the reorder quantity and the reorder point. In addition, since we will also be concerned with a system that has a single source and more than one inventory, we will need to determine the optimal allocation policy (i.e., how much of the reorder quantity to be sent to each inventory) and the optimal redistribution policy (how much to transfer from one inventory to the other). The formulation of our model will be described in detail in Chapter III, and the specific approaches to arriving at the optimal operating doctrine will be discussed in Chapter IV.

CHAPTER II

LITERATURE SURVEY

The literature that encompasses the discipline of scientific inventory control is vast by the most modest of estimates. Therefore, it will not be the purpose of this chapter to acquaint the reader with the entire field of inventory theory. There are many articles and books available that deal specifically with surveys of the important works in inventory theory. Most of these works have excellent bibliographies which provide points of departure for more extensive research into particular areas of inventory theory. One particularly good survey was done by Veinott (20). His survey deals mainly with works in the area of multi-installation and multi-product inventory theory. Another article by Iglehart (14), though somewhat brief in description, contains an excellent bibliography of important works in the field.

For the purposes of this study only those works that have bearing on our problem will be discussed. The problem will be treated as a single echelon, multi-installation problem with a single source of supply. In reality, the problem is actually a multi-product, single installation, single echelon problem that has a single source of supply. However, the products are related in such a manner as to permit conversion from one type product to another by additional processing. Since the alteration processes would take time and cost money, the product inventories can be looked upon as being in separate warehouses with redistributions between warehouses permitted. It can be seen that multi-product and multi-installation problems are similar in many respects.

The first articles published on multi-installation problems were by Clark and Scarf (6), (7). They dealt with a multi-echelon system where installation 1 received stock from 2 and 2 received stock from 3, etc. Using periodic review of on-hand inventory at the lowest level in the system, they developed a dynamic programming functional equation for the expected total cost for the system. They also consider the case where several installations are supplied from the same source. They compute the optimal policies at each installation separately and do not allow transshipments to take place between installations. However, they do take into consideration the allocation problems associated with a single source supplying more than one installation.

In a more recent article Bessler and Veinott (4) have considered a multi-installation system consisting of n installations, each stocking a common product and each drawing from the same source. The demands against an installation are stochastic, and any demand that cannot be met is passed along to the source to be filled. If it cannot be filled at the source, a shortage penalty is incurred, and the demand is backlogged. Again periodic review of on-hand inventory is used, and transshipment between installations is not allowed. Their ordering policy is to order sufficient stock at the beginning of each review period to bring the level of the system up to the quantity y. They then establish bounds on y, the upper bound being based on the assumption that the source has no stock on hand and the lower bound being based on the assumption that the source has an infinite supply on hand. The upper bound for a particular facility is based on the assumption that other facilities have no stock on hand. The lower bound for a particular facility is based on the assumption that the upper bounds are stocked at the other facilities. They then develop a procedure by which an approximation to the optimal solution can be found. The approximation to the number to be stocked at facility n is obtained by computing the optimal number to be stocked at facility n assuming that facility j $(j < n)$ stocks its lower bound. To compute the number to be stocked at facility k, they compute the optimal quantity to be stocked assuming that for $j > k$, the quantity stocked is the approximate quantity computed above and for $j < k$, facility j stocks its lower bound. It turns out that a base stock level characterized by vector y is the optimal policy provided the initial inventories are not too large.

More complicated multi-echelon models are examined by Gross in (9); however, his analysis is limited to single period models. Zangwill, in (21), discusses a class of multi-installation, multi-product models in which demands are deterministic. Although the analysis is much more sophisticated, the approach is similar to the classical economic lot-size problem. We will initially consider a deterministic model for our system using established doctrine for economic lot-size problems.

Hadley and Whitin (12) treat the case where one warehouse supplies an arbitrary number of installations and find optimal policies within a class of simple rules. Their approach fitted reasonably well the problem that we will be concerned with in this study. They assume demands to be stochastic according to known distributions and consider the possibility of transshipment between installations in the same echelon. They also assume the method of transactions reporting to a central control point and the placement of an order for Q units each time the system inventory position (the sum of the installation inventory positions) reaches a reorder point k. Their procedure is to develop an annual expected total cost function and separate functions for determining the optimal allocation rule and the optimal redistribution rule. The total cost function is then simplified by making further assumptions in order to reduce it to a form that is more easily minimized. The functions for optimal allocation rule and optimal redistribution rule are solved using dynamic programming.

Iglehart and Lalchandani (15) have considered the optimal dynamic policies for one warehouse feeding two installations. Here again, the method of periodic review is used, and the complicating factor of redistribution between installations is not considered. Their primary concern is the development of an optimal allocation policy for scarce resources between two installations.

An application of multi-installation techniques is made by Bessler (3). It involves a system in which the inventory items are high individual cost parts for submarines. The supply system consists of a depot and a specified number of submarine tenders. An item of unit cost C_2 can be stocked at any of the tenders and/or the depot. If the item is required by a submarine, it is supplied out of tender stock. If the tender is out of stock, the item is requisitioned from the depot and emergency shipped to the depot at a cost of C_0. If the depot stock is depleted, a system shortage cost, C_1, is incurred. Replenishment of the depot and tenders is made periodically by bringing the respective inventory levels up to critical values y_j. Using dynamic programming, they arrive at an optimal policy in the form of a critical vector of y_j's.

Shakun (18) describes a multi-warehouse, single-source system in which demands are normally distributed and transactions reporting is used. His objective is to answer the question of whether each warehouse should be operated individually on a self-sustaining basis or whether system wide control should be used with intershipments between warehouses permitted. To answer this question he performs a cost comparison between the two alternatives. He concludes that for items having a system-wide annual dollar usage of $1000.00 and over, system-wide operation is substantially better than independent warehouse operation. However, his analysis is only a comparison between two alternative systems, and it does not produce an optimal economic order quantity, Q, or reorder point, k. The values for these quantities are extracted from a model contained in another source.

Finally, Sadowski (17) and Frank (8) describe an application of multi-installation theory which they refer to as the assortment problem. The assortment refers to different sizes of steel beams that are stocked in order to meet demands for construction. If a demand for one size beam cannot be filled, it can be either backordered or a larger beam can be shortened (with the resultant waste of the unused portion) to fill the demand. Naturally, there are penalty costs involved for stockouts as well. The optimal assortment is the one which gives the greatest economy of steel and still satisfies all demands. A dynamic programming approach is used to determine the optimal assortment, and a discussion is given on the equivalence of the assortment problem to the economic lot-size problem.

CHAPTER III

DEVELOPMENT OF THE MODEL

The general approach to the formulation of the system model in subsequent sections of this chapter will be to first develop a model based on deterministic demands against the two inventories. This will provide a transitional base upon which to develop a model in which the demands are assumed to be stochastic in nature. Finally, a third model will be developed making use of approximations to facilitate ease of formulation and optimization. All three model formulations will result in single total cost objective functions to be minimized in the presence of the warehouse-space constraint function. Optimization procedures will be discussed in Chapter IV.

A Model with Deterministic Demands

The first model that we shall develop will be a representation of the system where the demands against each inventory are known with certainty and are constant over time. It is felt that this approach will provide a framework upon which can be constructed the more complex stochastic demand model. While the deterministic model is not an accurate depiction of the true system, it may be representative enough to provide insight into the true behavior of the system.

The general sequence in the development of the deterministic model will be as follows:

(1) Develop the cost function, K, for a system in which

redistribution between inventories and assembly of product 2 units are assumed to be free and instantaneous.

(2) Develop the total cost function, TC, for the system when the assumption of free and instantaneous redistribution is dropped.

(3) Develop the allocation decision function and incorporate it into the total cost function.

(4) Develop the warehouse constraint function.

Development of the Cost Function with Free Redistribution

Under the assumption of free and instantaneous redistribution and product 2 assembly the two inventories may be treated as a single inventory, and the formulation process is greatly simplified. The costs that must be contained in the cost expression, K, are simply the procurement, inventory, and system shortage costs. For this problem, K will represent the average annual cost for the system.

The procurement cost for the system consists of three parts: (1) a fixed cost A incurred each time an order is placed; (2) the unit cost of the item, $C(Q)$; and (3) transportation costs from factory to warehouse. If a quantity Q units is ordered each time the inventory position of the system reaches a value, k, and the system rate of demand is D units per year, then the number of orders placed per year must average to D/Q over the long term. Thus, the average fixed procurement cost per year is AD/Q. The average yearly variable cost per year will be $DC(Q)$. However, it will be assumed for this problem that the unit price of an item is independent of the quantity ordered. Consequently, $C(Q)$ will be a constant, C, and DC will be independent of Q and the reordering rule. It will also be assumed that the transportation costs (J dollars per item) between factory and warehouse are independent of the quantity ordered. Consequently, the average yearly transportation costs, JD, will also be independent of Q and the reordering rule. Thus, the average annual procurement cost will consist only of the fixed cost AD/Q. A more detailed development of the above can be found in (11).

We must next consider the inventory carrying costs for the system. Here we will make the assumption that the instantaneous rate at which inventory carrying costs are incurred are proportional to the investment in inventory at that point in time. It will also be assumed that I, the constant of proportionality, is known approximately and is constant with respect to time. It is measured in terms of dollars per year per dollar of inventory invested. The symbol, I, will be referred to as the carrying charge and will be used as a simplifying approximation of the cumulative and instantaneous effects upon carrying costs by such factors as insurance, taxes, breakage, pilferage, warehouse maintenance, and lost opportunity. Lost opportunity refers to the amount of income that is lost by having capital tied up in inventory rather than in some other investment that yields a return. The carrying charge is explained more thoroughly in Chapter I of (11).

It has been shown by Hadley and Whitin (11) that the inventory carrying costs per year must be IC times the average inventory. However, in determining the average inventory we must contend with such factors as integrality of demand, finite production rate, and the possibility of stockouts. It will be assumed that all demands will be met from the stock on hand and that any demand which cannot be met because of insufficient stock on hand will be backordered until sufficient stock is available. Lost sales will not be considered since it is seldom optimal to incur lost sales in a deterministic demand system (11).

We assume that Q is sufficiently large to treat the demand as continuous. In other words, the demand will be assumed to be continuous and Q^*, the optimal order quantity, will be rounded off to the nearest integer. Although in reality demands are for integral numbers of units, computations for integrality of demand may be neglected since their effect on Q^* will be negligible.

Since our inventory system deals with a factory and a factory warehouse, it is logical to assume that the products are delivered to the warehouse as they are produced. The rate of production will be assumed finite and will be measured in terms of P units per year. It will also be assumed that the rate of production, P, is greater than the rate of demand, D. An earlier assumption was made that all demands would be met, but that stockouts would be possible and demands would be backordered in the event of stockouts. In computing the contributions of carrying costs and backorders to the cost expression, K, both factors, finite production rate and backorders, must be considered together. Procurement costs will not be affected and will remain DA/Q.

It will be assumed that the cost of a backorder has the form $\pi + \hat{\pi}t$ where t is the length of time for which a backorder exists. The symbol π will represent a fixed cost associated with each backorder and $\hat{\pi}$ a cost proportional to the length of time for which the backorder exists.

Let s be the number of backorders on the books when the first increment of products arrives from the factory. During periods when the products are being produced in the factory, there will be a net rate of inflow (P-D) of units into the warehouse. During the period when the factory is not producing there is a net rate of outflow, D, of units from the warehouse. The situation is illustrated geometrically in Figure 2. The length of time required to produce a lot, Q, is $T_1 + T_2 = Q/P$. Since the on-hand inventory in the warehouse reaches its maximum value just as production is stopped at the factory, the maximum on-hand inventory is $T_2(P-D) = (Q/P-T_1)(P-D)$. The time required to fill all backorders and bring the on-hand inventory level to zero is $T_1 = s/(P-D)$. Therefore, the maximum on-hand inventory can be written

$$T_2(P-D) = Q(1-D/P) - s. \tag{1}$$

The time required to reduce the on-hand inventory in the warehouse to zero is

$$T_3 = Q/D(1-D/P) - s/D. \tag{2}$$

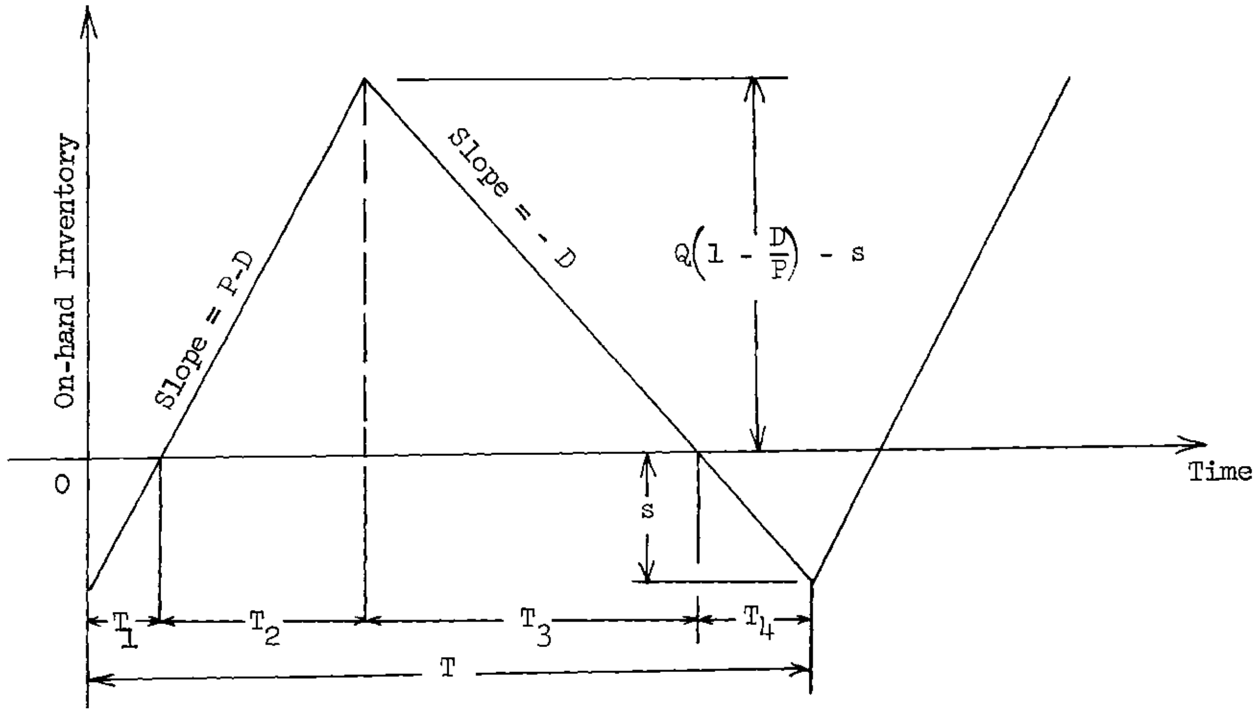

Figure 2. Geometric Representation of Deterministic System

The time required to incur s backorders is $T_4 = s/D$. The length of a cycle is $T = T_1 + T_2 + T_3 + T_4 = Q/D$. However, the inventory carrying costs per cycle are based on times T_2 and T_3 since the inventory level is zero for times T_1 and T_4. Therefore, the inventory carrying costs per cycle are

$$IC \left\{ \int_0^{T_2} (P-D)t \; dt + \int_0^{T_3} \left\{ \left[Q(1 - \frac{D}{P}) - s\right] - Dt \right\} dt \right\}$$

$$= IC \left\{ \frac{1}{2(P-D)} \left[Q(1 - \frac{D}{P}) - s\right]^2 + \frac{1}{2D} \left[Q(1 - \frac{D}{P}) - s\right]^2 \right\}$$

$$= IC \frac{P}{2D(P-D)} \left[Q \; (1 - \frac{D}{P}) - s\right]^2. \qquad (3)$$

The average annual cost of holding inventory is then found by dividing by the time, T, for completion of one cycle, i.e.,

$$IC \frac{P}{2Q(P-D)} \left[Q \; (1 - \frac{D}{P}) - s\right]^2. \qquad (4)$$

We must now determine the average annual cost of backorders. As explained earlier, the backorder cost has the form $\pi + \hat{\pi}t$. Therefore, the backorder cost per cycle is

$$\pi s + \hat{\pi} \left\{ \int_0^{T_1} \left[s - (P-D)t \right] dt + \int_0^{T_4} Dt \, dt \right\}$$

$$= \pi s + \hat{\pi} \left\{ \frac{1}{2} \frac{s^2}{(P-D)} + \frac{s^2}{2D} \right\}$$

$$= \pi s + \frac{\hat{\pi} P s^2}{2D(P-D)}. \tag{5}$$

The average annual cost for backorders is then

$$\frac{1}{Q} \left[\pi D s + \frac{1}{2} \frac{\hat{\pi} P s^2}{(P-D)} \right]. \tag{6}$$

The average annual variable cost, K, which includes the costs of ordering, holding inventory, and backorders then becomes

$$K = \frac{AD}{Q} + IC \frac{P}{2Q(P-D)} \left[Q(1 - \frac{D}{P}) - s \right]^2$$

(equation continued)

$$+ \frac{1}{Q}\left[\pi Ds + \frac{1}{2}\,\frac{\pi P s^2}{(P-D)}\right]. \qquad (7)$$

We have now completed the development of a cost function, K, for a system in which redistribution between inventories and assembly of product 1 are assumed to be free and instantaneous. The function, K, will be one part of the total cost model of the deterministic system.

Development of the Cost Function with Redistribution, Product Shortage, and Allocation Costs

We will now turn our attention to the development of a total cost function, TC, for the system when the assumption of free and instantaneous redistribution is dropped and the allocation decision is taken into account. The assumption of free and instantaneous assembly of product 1 prior to transportation to the warehouse will also be dropped.

As stated earlier, when the inventory position for the system as a whole reaches a point, k, then an order for Q units will be sent to the factory. It was also stated earlier that demands against each inventory were independent and not necessarily at the same rate. However, since we are operating under the assumption of deterministic demands, it should be possible to develop an allocation decision rule that should be fixed with each order. In other words, the order quantity can be broken into two component fixed quantities, q_1 and q_2, which represent the amounts allocated to the product 1 and product 2 inventories respectively.

If "J" is the cost per unit to transport unassembled components from factory to warehouse and if "a" is the cost per unit of assembling components into finished products then the annual transportation costs involved in shipping units from factory to warehouse are D_2J for the product 2 inventory and $D_1(J + a)$ for the product 1 inventory where D_1 and D_2 are the rates of demand for products 1 and 2 respectively. The total transportation cost for the system is then

$$V = D_2J + D_1 (J + a). \qquad (8)$$

Since demands against the system are deterministic, it can be shown that redistributions between the two inventories can never be an optimal course of action. The intuitive argument that follows will be sufficient to show that this is the case. Assume for a given period that the reorder quantity Q is not allocated on the basis of demand against each inventory as indicated in the previous paragraph; e.g., suppose q_1' and q_2' are such that inventory 1 is depleted well in advance of inventory 2 and the decision is made to redistribute a quantity x_{21} units from 2 to 1. This would entail assembling x_{21} units of product 2 components to convert them to units of product 1. Assume that redistribution of x_{21} units will cause both inventories to be depleted at the same time. It also seems logical to assume that this conversion process would have to take place at the factory. Thus, as a minimum, the following costs per unit would be incurred in converting units of product 2 to product 1: (a) "J" dollars per unit to transport them to the factory from inventory 2; (b) "a" dollars per unit to assemble the components into units of product 1; and (c) "J" dollars per unit to ship the assembled units from the factory to inventory 1. Therefore, the cost of redistributing x_{21} units would be $x_{21}(2J + a)$ and the total transportation and conversion costs for the period would be $v' = q_1'(J + a) + q_2' J + x_{21} (2J + a)$. On the other hand, since it is possible, in a deterministic system, to predict with certainty the individual inventory levels at any future point in time, the values for q_1 and q_2 can be adjusted to the point where both inventories are depleted simultaneously. Therefore, if we let $q_1 = q_1' + x_{21}$ and $q_2 = q_2' - x_{21}$, the transportation costs for the period will be $v = q_1(J + a) + q_2 J = q_1'(J + a) + q_2' J + x_{21}a$. We can now see that $v < v'$ by the amount $2x_{21}J$. It is obvious that v' would be even greater in value for redistribution from 1 to 2 since we would have the additional conversion costs of disassembly of product 1 units. Therefore, it will be concluded that redistribution can be disregarded for the deterministic model. This, however, does not preclude the necessity to include the stockout costs for the individual inventories.

In step 1 we computed the added stockout costs for the entire system being out of stock. The same procedure will now be used for each of the inventories individually. As in the system backorders case the costs of backorders for the product 1 and product 2 inventories will take the forms $\pi_1 + \hat{\pi}_1 t_1$ and $\pi_2 + \hat{\pi}_2 t_2$ respectively where t_1 and t_2 are the respective lengths of time for which backorders exist. Again, the symbols π_1 and π_2 will represent the fixed costs associated with each backorder and $\hat{\pi}_1$ and $\hat{\pi}_2$ the costs proportional to the lengths of time for which the backorders exist. In addition the following notation will apply:

s_1 = the number of backorders for product 1 when the first increment of the q_1 units allocated to the product 1 inventory arrives.

s_2 = the number of backorders for product 2 when the first increment of the q_2 units allocated to the product 2 inventory arrives.

P_1 = rate of production of product 1.

P_2 = rate of production of product 2.

D_1 = rate of demand for product 1.

D_2 = rate of demand for product 2.

It should be noted that $q_1 + q_2 = Q$, $s_1 + s_2 = s$, $P_1 + P_2 = P$, and $D_1 + D_2 = D$.

Following the same procedure used in step 1 for the system as a whole, the backorders costs per cycle are

$$\pi_1 s_1 + \frac{\hat{\pi}_1 \, P_1 \, s_1^2}{2 \, D_1 \, (P_1 - D_1)} \tag{9}$$

for the product 1 inventory, and

$$\pi_2 s_2 + \frac{\hat{\pi}_2 P_2 s_2^2}{2 D_2 (P_2 - D_2)} \tag{10}$$

for the product 2 inventory. The average annual costs of backorders are

$$B_1 = \frac{1}{q_1}\left[\pi_1 D_1 s_1 + \frac{1}{2}\frac{\pi_1 P_1 s_1^2}{(P_1 - D_1)}\right] \tag{11}$$

for the product 1 inventory, and

$$B_2 = \frac{1}{q_2}\left[\pi_2 D_2 s_2 + \frac{1}{2}\frac{\pi_2 P_2 s_2^2}{(P_2 - D_2)}\right] \tag{12}$$

for the product 2 inventory.

By combining equations (7), (8), (11), and (12) we get the following total cost model for the system:

$$TC = K + V + B_1 + B_2. \tag{13}$$

This function, when minimized subject to the warehouse-space constraint, will provide the optimal operating policy for the deterministic system.

Development of the Warehouse Constraint Function

The final step in the formulation of the deterministic model is the development of the warehouse constraint function. We know that the maximum on-hand inventory for the system is $Q(1 - D/P) - s$ from equation (1). Likewise, the maximum on-hand inventories for the product 1 and product 2 inventories taken separately are $q_1(1 - D_1/P_1) - s_1$ and $q_2(1 - D_2/P_2) - s_2$ respectively. Let w_1 and w_2 be the amounts of warehouse space per unit consumed by product 1 and product 2 respectively. If W is the total amount of warehouse space available for both inventories, then the warehouse constraint is

$$w_1 \left[q_1(1 - D_1/P_1) - s_1 \right] + w_2 \left[q_2(1 - D_2/P_2) - s_2 \right] \leqq W. \quad (14)$$

Our model of the deterministic system is now complete. To obtain the optimal operating policy for the inventory system we minimize the total cost function, equation (13), subject to the inequality (14). The development of the optimal operating policy will be discussed in Chapter IV.

A Model with Stochastic Demands

We now wish to develop a model in which we are concerned with demands on the system that cannot be predicted with certainty but instead must be described probabilistically. In developing this model the assumptions will be the same as for the deterministic model with the exception that the demands against each inventory can be characterized by Poisson distributions with mean rates of demand D_1 and D_2 (for product 1 and product 2 respectively) which do not change with time. Here again the assumption will be made that the units of products will be demanded one at a time.

The general procedure for the development of the stochastic model will be similar to that used in developing the deterministic model although the techniques involved will be considerably different. The procedure will be as follows:

(1) Develop the cost function, K, for a system in which redistribution between inventories and assembly of product 1 are assumed to be free and instantaneous.

(2) Develop the total cost function, TC, for the system when the assumption of free and instantaneous redistribution is dropped.

(3) Develop a model for finding the optimal redistribution rule. This will be a separate model from the total cost model for the system.

(4) Develop a model for finding the optimal allocation decision rule. This will also be a separate model from the total cost model.

(5) Develop the warehouse constraint function.

Having developed the model, the general procedure for its optimization will be to first find the optimal values for Q and k by minimizing the total cost function (developed in step 2 above) subject to the warehouse constraint (developed in step 5). This value of Q can then be used in the redistribution and allocation functions (developed in steps 3 and 4 respectively) which are minimized to determine the optimal redistribution quantity and optimal allocation rule respectively.

Development of the Cost Function for Free and Instantaneous Redistribution

In step 1 for the stochastic model the assumption of free and instantaneous redistribution and assembly of product 1 enables us to treat the two inventories as a single inventory system. Here again we must develop a function for K, the average annual cost, which takes into account procurement, inventory carrying, and system shortage costs.

The procurement costs for the stochastic model will be the same as for the deterministic model with $D = D_1 + D_2$ now representing the mean system demand. Note that since the demands against the two inventories are Poisson distributed, the system demand is also Poisson distributed (8). Since the unit cost, C, of a unit and transportation cost, J, per unit are independent of the quantity ordered, the average annual procurement costs will again consist only of the fixed cost, AD/Q.

The remainder of K will be developed using the same procedure as that used by Hadley and Whitin in (12) to develop their cost function, K(Q,k). The effects of the assumption of finite production rate, which is made here but not for the Hadley and Whitin model, are nullified by making inventory position the parameter for inventory level rather than on hand inventory. In other words, inventory level will be measured in terms of inventory position (on hand plus on order minus backorders) instead of only on-hand inventory. The situation is illustrated graphically by Figure 3. The reorder rule for the system will again be to order a fixed quantity, Q, each time the inventory position of the system reaches a level k. Since the assumption has been made that units are demanded one at a time, the inventory position of the system can only take on one of Q different values, i.e., $k + 1, k + 2, \ldots, k + Q$. Assuming that the inventory system has reached a steady state of operation and the mean rate of demand, D, for the system remains constant over time, then

$$DP(j + 1) = DP(j) \quad (j = k + 1, \ldots, k + Q - 1)$$

where $P(k + i)$ is the probability that the inventory position of the system is valued at $k + i$. It then follows that

$$P(k + Q) = P(k + Q - 1) = \ldots = P(k + 1) = 1/Q \qquad (15)$$

thus indicating that the steady state probability distribution is rectangular for Poisson demands (12).

The inventory holding costs will be based on the average quantity on hand plus on order. This can be done without affecting Q and k since the difference between average on-hand inventory and average

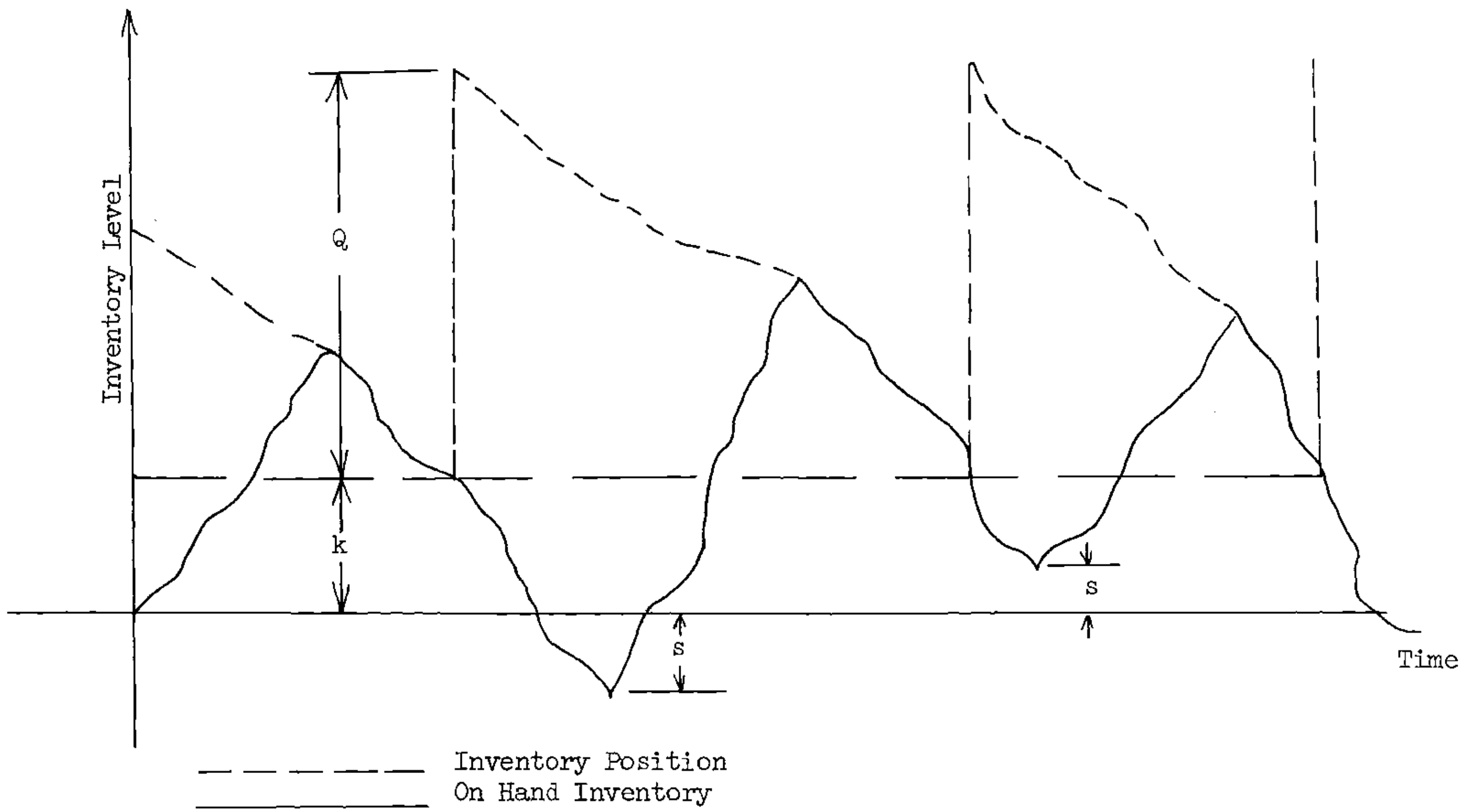

Figure 3. Graphical Representation of Stochastic System

on-hand plus on-order inventory is a constant which is equal to the average amount on order. This assumes that backorders are negligible. Since the inventory position of the system has the value, $k + i$ $(i = 1, \ldots, Q)$, with probability $1/Q$, the average inventory position is

$$\frac{1}{Q}\sum_{i=1}^{Q}(k + i) = \frac{Q}{2} + k + \frac{1}{2} \tag{16}$$

Let F equal the expected on hand plus on order inventory and let B equal the expected number of backorders at any point in time. Since the inventory position of the system is equal to the amount on hand plus on order minus the number of backorders, then

$$F = Q/2 + k + 1/2 + B, \tag{17}$$

and the expected carrying charge per year is ICF where I and C are as defined for the deterministic model.

We must now determine the expected system shortage costs. As in the deterministic model the cost of a backorder will take the form $\pi + \hat{\pi}t$ and the expected annual cost of the system shortages is π times the expected number of shortages per year, E, and $\hat{\pi}$ times the expected unit years of shortage which is merely the expected number of backorders, B (times 1 year). To avoid confusion between the quantities E and B the definition of the terms will be described in detail. Since we have assumed that any shortage incurred is backordered, E is also the expected number of backorders incurred per year. Note that we have stated that E is the expected number of backorders incurred in a year, not the expected number of backorders at any point in time.

Suppose that we start observing the inventory system at some arbitrary point in time and assume that in a time ζ since the observation was begun the system goes through n complete cycles. Suppose further that backorders which are incurred during a cycle are labeled consecutively (1, 2, . . . , b) and that the amount of time (measured in terms of years) for which each backorder exists is labeled t_{ij} ($j = 1, 2, \ldots, b$) where i represents cycle i. Then the unit years of shortage incurred during cycle i are $\Delta_i = \sum_{j=1}^{b} t_{ij}$. Let ξ_i be the total number of backorders incurred during cycle i. As $\zeta \to \infty$, it must be true that $n \to \infty$ and by definition (11)

$$B = \lim_{\zeta \to \infty} \frac{\sum_{i=1} \Delta_i}{\zeta}, \quad E = \lim_{\zeta \to \infty} \frac{\sum_{i=1} \xi_i}{\zeta}$$

are the expected unit years of shortage incurred per year and the expected number of backorders incurred per year. If the number of shortages were always a constant B, then the unit years of shortage incurred per year would be B. Thus, B is the expected number of backorders on the books at any point in time. Note that since B is a function of Δ_i, the term "backorder" implies a shortage over time relationship. A more detailed discussion of the above can be found in Chapter 4 of (11).

We shall now proceed with the development of specific expressions for E and B to fit our model. To determine the expected number of shortages per year, E, we must first compute P_{out}, the probability that the system is out of stock at any instant of time. Assuming that τ, the procurement lead time (time between placing an order and arrival of the first increment of an order), is constant, and if the system is in state $\jmath$ at time t, then everything on order at time t will have arrived by time $t + \tau + Q/P$ where P is the constant rate of production. The assumption will be made that τ is small in comparison with the length of a cycle, T. By making this assumption we prevent the possibility of overlapping orders; i.e., the placing of a second order before or during the arrival of the previous order at the warehouse. In addition, a relatively small τ will keep the number of backorders incurred small in comparison with the average on-hand inventory. This enables us to use the average on-hand plus on-order inventory to calculate carrying costs instead of just the average on-hand inventory. Since there will be no overlapping orders, then nothing ordered later than time t can arrive by $t + \tau$. Let $p(x, D\tau)$ be the Poisson density, that is, the probability that x units are demanded in time τ where D is the mean rate of demand. Then the probability that no units are on hand at time $t + \tau$ when the system is in state $\jmath$ at time t is the probability that $\jmath$ or more units are demanded in time τ; i.e.,

$$P(j, D_T) = \sum_{x=j}^{\infty} p(x, D_T) \tag{18}$$

Since j can take on the values $k + 1$, $k + 2$, . . . , $k + Q$, the probability that the system is out of stock at any time is found by averaging over all possible states j; i.e.,

$$P_{out} = \frac{1}{Q} \sum_{j=k+1}^{Q+k} P(j, D_T) = \frac{1}{Q}\left[\sum_{j=k+1}^{\infty} P(j, D_T) - \sum_{j=Q+k+1}^{\infty} P(j,D_T)\right]. \tag{19}$$

From property 9 of (8) we know that

$$\sum_{j=k+1}^{\infty} P(j, D_T) = D_T\ P(k, D_T) - k\ P(k + 1, D_T) \tag{20}$$

and

$$\sum_{j=Q+k+1}^{\infty} P(j, D_T) = D_T\ P(Q + k, D_T) - (Q + k)\ P(Q + k + 1, D_T) \tag{21}$$

Therefore it follows that

$$P_{out} = \frac{1}{Q}\left[g(k) - g(Q + k)\right] \tag{22}$$

where

$$g(u) = \sum_{x=u+1}^{\infty} P(x, D_\tau) = D_\tau P(u, D_\tau) - u P(u + 1, D_\tau). \tag{23}$$

The expected number of shortages, E, is the mean rate of demand times the probability that the system has no stock on hand at any instant of time; i.e.,

$$E = DP_{out} \tag{24}$$

We must next determine B, the expected number of backorders at any point in time (expected unit years of shortage incurred per year). If the system is in state j at time t, the expected number of backorders at time $t + \tau$ is

$$\sum_{x=j+1}^{\infty} (x - j)\, p(x, D_T).$$

The expected number of backorders at any time is then found by averaging over the states j; i.e.,

$$B = \frac{1}{Q} \sum_{j=k+1}^{Q+k} \sum_{x=j+1}^{\infty} (x - j)\, p(x, D_T) \tag{25}$$

From properties 7, 9, and 21 of (13) we know that B can be written as follows:

$$B = h(k) - h(Q + k) \tag{26}$$

where

$$h(u) = \alpha(u)\, P(u, D_T) + \frac{D_T(D_T - 1)}{Q}\, P(u - 1, D_T) - \frac{(D_T)^2}{2Q}\, P(u - 2, D_T) \tag{27}$$

and

$$\alpha(u) = \frac{1}{Q}\left[D_T(1 - u) + \frac{u(u + 1)}{2}\right] \tag{28}$$

We have now completed the development of all the terms in the cost function K which may be written as

$$K = \frac{DA}{Q} + IC\left[\left(\frac{Q}{2} + k + \frac{1}{2}\right) + B\right] + \pi E + \hat{\pi} B \tag{29}$$

We are now ready to develop the total cost function, TC, in which redistribution costs are included.

Development of the Total Cost Function

The assumption was made that one product could be converted to another if redistribution between inventories was necessary. By necessary we mean that redistributions are made only when there is likely to be a need for them prior to the next allocation of products from the factory. We will therefore consider redistribution as a means of reducing possible shortage costs between allocations.

In this section we will be concerned with developing redistribution cost expressions to be added to K in the formulation of the total cost function, TC. The questions of when redistribution should be considered and what quantities should be redistributed will be dealt with in detail when we develop the redistribution rule in a later section. They also must be considered in the development of the redistribution cost expressions. The question of when a redistribution should be considered can be answered by observing that a redistribution should be considered when the on-hand inventory for any inventory becomes dangerously low in the period between allocations. Let the time required to transfer stock from inventory i to inventory j (including the conversion process) be called the redistribution lead time, τ_{ij}, where i, j = 1, 2 and $i \neq j$. Now the on-hand inventory will be considered to be dangerously low if during the lead time τ_{ij} the probability of incurring one or more backorders reaches a critical value, α_j. The critical values, α_j, will be assumed to be known. Each α_j determines an on-hand inventory level, M_j, for inventory j such that when the on-hand inventory falls to a level M_j, inventory j is said to be at its critical probability level. The M_j's will be referred to as "M levels" for the respective inventories and when reached will initiate consideration for redistribution.

It will not always be desirable to consider redistribution when an M level is reached since the next allocation may be scheduled to begin arriving before a redistribution could be effected. Thus τ_{ij} will be the time between making the decision to redistribute from i to j and the arrival of the units at j, and no redistribution will be considered when inventory j reaches M_j at a time less than τ_{ij} from the arrival of the first increment of the next factory allocation.

Let time T_{cj} be the measure of time from the arrival of the first increment of the previous allocation to a time τ_{ij} prior to the arrival of the first increment of the next allocation. The times, T_{cj} ($j = 1, 2$), will be known as cutoff times and are illustrated along with the M levels in Figure 4. From Figure 4, we see that when inventory j reaches its M level before its cutoff time, a redistribution will be considered. Note that it will only be considered and may or may not be made depending on other factors and considerations. It will be assumed that τ, the production lead time is greater than τ_{ij}. This is logical to assume since the production set-up, assembly, and transportation times that comprise τ would realistically be greater than the assembly, disassembly, and transportation times that comprise τ_{ij}. This assumption is necessary to avoid the situation where an order is placed and begins arriving all within the redistribution lead time. Such a situation would nullify the usefulness of the redistribution.

As stated earlier an M level is selected for each inventory by specifying the critical probabilities, α_j($j = 1, 2$), that the inventory will have one or more backorders in the redistribution lead time, τ_{ij}. Therefore M_j is the smallest M such that

$$P(M_j + 1, D_j \tau_{ij}) \leq \alpha_j \tag{30}$$

where D_j is the mean rate of demand against inventory j.

We must now compute the expected costs of redistribution and

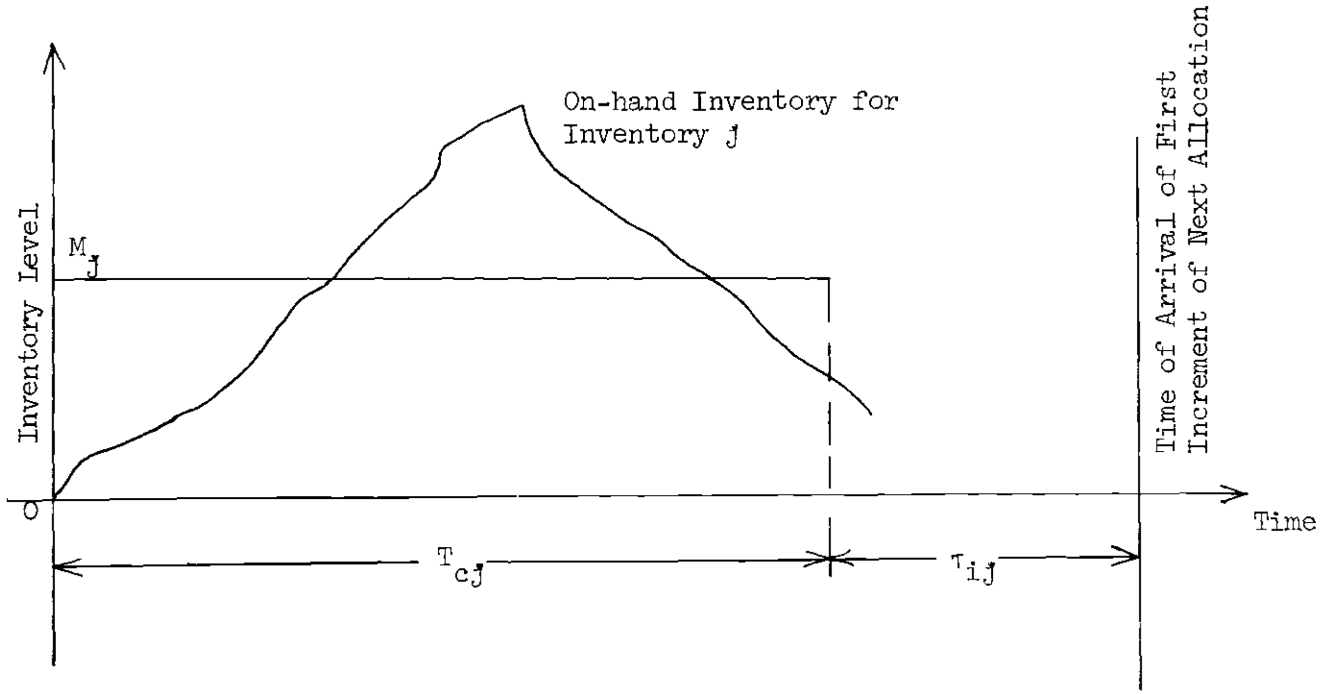

Figure 4. Graphical Representation of M Levels and Cutoff Times

inventory shortages for a given set of M levels. We will make the same assumptions as do Hadley and Whitin (12). First let R_{ij} (i, j = 1, 2 and i $\neq$ j) be the expected costs associated with hitting an M level and assume that they are known quantities. They include the expected cost of redistribution from inventory i to inventory j plus the cost of making a redistribution calculation (if any). The act of specifying the R's is analogous to specifying the shortage costs but with fewer "intangibles" involved. A second assumption to be made is that a given M level is reached only once in a period (cycle). The instances of more than one redistribution in a single period would be sufficiently rare as to make their effect on Q and k negligible.

Assume that the maximum on hand inventory levels are achieved at the termination of production for a particular order. Let T_{mj} be the time between receipt of the first increment of an allocation by inventory j and the achievement of g_j, the maximum level of inventory j for that period. Also assume that $g_j > M_j$ and $T_{cj} > T_{mj}$. Note that T_{mj} is a function of Q_j, the quantity of Q allocated to inventory j; i.e., $T_{mj} = Q_j/P_j$ where P_j is the known and constant rate of production for product j. Also note that the value of g_j is a function of T_{mj}, the rate of demand, and the number of backorders at time T_D, the time of arrival of the first increment of an allocation at inventory j. Thus g_j and T_{mj} are random variables.

For a given g_j consider the probability that M_j will be reached before the cutoff time, T_{cj}. The cutoff time if $T_{cj} = T - \tau_{ij}$ (i, j = 1, 2 and i $\neq$ j) where T is the time of arrival of the first increment of the next allocation. If lead times are constant, the probability that the next allocation arrives between T and dT is the probability that $Q - 1$ units are demanded between T and $T + dT$. Here the assumption is made that the production lead times for both inventories are equal and consequently T is the same for both inventories. The probability that the Q^{th} demand occurs in $T + dT$ can be thought of as the probability of X and Y where X is the event that $Q - 1$ demands occur between T_0 and T and Y is the event that a demand occurs between T and dT. Therefore, for a Poisson process in which Y is independent of X, $p(X\ Y) = p(X)\ p(Y)$ where

$$p(X) = p(Q - 1, DT),\ p(Y) = DdT. \tag{31}$$

Thus the probability that the Q^{th} demand occurs in $T + dT$ is

$$\frac{D(DT)^{Q-1} e^{-DT}}{(Q-1)!} dT. \tag{32}$$

The density function for T is therefore

$$f(T) = \frac{D(DT)^{Q-1} e^{-DT}}{(Q-1)!} \tag{33}$$

If τ_{ij} $(i, j = 1, 2 \text{ and } i \neq j)$ is known and is constant, then the density function of T_{cj} is $f(T_{cj} + \tau_{ij})$.

The probability that on hand inventory at j reached M_j before a given T_{cj} is the probability that at least $g_j - M_j$ units are demanded in time $T_{cj} - T_{mj}$, i.e.,

$$P[g_j - M_j, D_j(T_{cj} - T_{mj})].$$

Hence for a specific g_j, the probability of reaching M_j before the cutoff time, T_{cj}, is

$$\int_0^\infty P[g_j - M_j, D_j(T_{cj} - T_{mj})]\ F(T_{cj} + \tau_{ij})\ dT_{cj}. \tag{34}$$

$$(i, j = 1, 2 \text{ and } i \neq j)$$

Let $p(g_j)$ be the probability of g_j and let D/Q be the average number of periods per year. Then, after averaging over g_j, the expected annual cost of allowing inventory j to reach M_j and effecting redistribution is

$$Z_j^{(1)} = \frac{R_{ij}D}{Q} \sum_{g_j=M_j+1}^{\infty} p(g_j) \int_0^{\infty} P[g_j - M_j, D_j(T_{cj} - T_{mj})]F(T_{cj}+\tau_{ij})dT_{cj} \quad (35)$$

$$(i, j = 1, 2 \text{ and } i \neq j)$$

In the development of their model Hadley and Whitin ignore the situation where no redistribution is made when an M level is reached and backorders occur before the next allocation. It is felt that this situation can have a significant effect on Q and k for this problem since the redistribution costs from inventory 1 to inventory 2 could possibly be prohibitively high. Therefore, we must compute the expected annual cost of allowing inventory j ($j = 1, 2$) to reach M_j and allowing backorders to incur instead of effecting redistribution.

Expression 34 is the probability of reaching M_j before the cut-off time T_{cj}. Given that M_j has been reached before T_{cj}, if the number of demands by T_{cj} is $g_j - M_j + x$ ($x = 1, 2, \ldots$), then the probability of incurring one or more stockouts in time τ_{ij} is $P(M_j - x + 1, D_j \tau_{ij})$. Therefore, the probability that inventory j will be out of stock at time $T_{cj} + \tau_{ij}$ is

$$P_{j,\text{out}} = \sum_{x=1}^{\infty} p[g_j - M_j + x, D_j(T_{cj} - T_{mj})]P(M_j - x + 1, D_j\tau_{ij}) \ldots (36)$$

$$\cdots \sum_{g_j=M_j+1}^{\infty} p(g_j) \int_0^{\infty} P[g_j - M_j, D_j(T_{cj} - T_{mj})] f(T_{cj} + \tau_{ij})\, dT_{cj}$$

and

$$E_j = D_j P_{j,\text{out}} \tag{37}$$

where E_j is the expected number of shortages in inventory j per year.

Next we shall evaluate the expected number of backorders for inventory j. If inventory j is in state $(M_j - x)$ at time T_{cj}, then the expected number of backorders at time $T_{cj} + \tau_{ij}$ is

$$\sum_{x=1}^{\infty} x\ p(M_j, D_j\ \tau_{ij}) \tag{38}$$

Assume $M_j - x$ is never negative at T_{cj}. The expected number of backorders at any time is then found by averaging over all states $(M_j - x)$ and multiplying by the probability of reaching M_j before T_{cj}; i.e.,

$$B_{ij} = \frac{1}{M_j} \sum_{M_j - x = 0}^{M_j} \sum_{x=1}^{\infty} x\, p(M_j,\, D_j \tau_{ij}) \cdots \tag{39}$$

$$\cdots \sum_{g_j = M_j + 1}^{\infty} p(g_j) \int_0^{\infty} P[g_j - M_j,\, D_j(T_{cj} - T_{mj})] f(T_{cj} + \tau_{ij})\, dT_{cj} .$$

Letting the cost of a backorder for inventory j have the form $\pi_j + \hat{\pi}_j \tau$ and employing the same logic used earlier in the development of K, we find that the expected annual cost of allowing inventory j to reach M_j and incurring backorders instead of redistribution is

$$Z_j^{(2)} = \pi_j\, E_j + \hat{\pi}_j\, B_j\, . \tag{40}$$

Combining equations 35 and 40 we arrive at the expected yearly cost of redistribution and inventory stockouts; i.e.,

$$Z = \sum_{j=1}^{2} [Z_j^{(1)} + Z_j^{(2)}] \tag{41}$$

By combining this result with the cost expression, K, we find that the expected total cost is

$$TC = K + Z \tag{42}$$

where K is given by equation 29 and Z by equation 41. This function must be minimized to determine Q and k.

Development of the Redistribution Rule Model

Now that we have a function for determining Q and k, we must turn our attention to the development of a function that will enable us to find how much to redistribute when redistribution becomes necessary. Assuming one of the inventories has reached its M level, the optimal redistribution policy is determined by minimizing the expected costs for the system over the time until the next allocation.

Assume inventory j has reached level M_j. Let C_{ij} represent the cost per unit of converting a unit of inventory i to a unit of inventory j. Assume that the C_{ij}'s are fixed and that $C_{ij} \neq C_{ji}$ (i, j = 1, 2 and i $\neq$ j). If inventory i redistributes $x_{ij} \geq 0$ units to j, the cost of conversion will be $C_{ij}\ x_{ij}$.

In addition to conversion costs, we must also consider shortage costs and costs arising when inventories reach their M levels later in the period. Let y_i be the on-hand inventory at i at the time inventory j reaches M_j. Then the on-hand inventory at i immediately after redistribution will be $y_i - x_{ij}$. Let t be the time since the previous allocation when j reaches M_j and let r be the number of units demanded in the entire system since the last allocation. Let $f_r(t_a)dt_a$ be the probability that the next allocation is made between t_a and dt_a, where time t_a is measured from the time j reaches M_j. Then the probability that the Q^{th} demand occurs in $t_a + dt_a$ can be thought of as the probability of X and Y where X is the event that Q - r - 1 demands occur between t and t_a and Y is the event that a demand occurs between t_a and dt_a. Therefore, for a Poisson process in which Y is independent of X, p(X Y) = p(X) p(Y) where $p(X) = p(Q - r - 1, Dt_a)$ and $p(Y) = Ddt_a$. Therefore, it follows that the probability of the Q^{th} demand occurring in $t_a + dt_a$ is

$$\frac{D(Dt_a)^{Q-r-1}\, e^{-Dt_a}}{(Q-r-1)\,!}\, dt_a \tag{43}$$

The density function for t_a is therefore the Erlang density,

$$f_r(t_a) = \frac{D(Dt_a)^{Q-r-1}\, e^{-Dt_a}}{(Q-r-1)\,!} \tag{44}$$

If we let t_i represent the time, measured from the time inventory j reaches M_j, until the cutoff time, T_{ci}, then the probability density function for t_i is $f_r(t_i + \tau_{ji})$, since τ_{ji} is a constant.

Assume that inventory i is not allowed to ship so much to j that $y_i - x_{ij} \leq M_i$. Then for a given t_i the probability that inventory i reaches M_i before the cutoff time is $P(y_i - x_{ij} - M_i, D_i t_i)$. Therefore, the expected cost if inventory i reaches M_i before the cutoff time is

$$\Gamma_i^{(1)} (y_i - x_{ij}, t_i) = R_{ji} \int_0^\infty P(y_i - x_{ij} - M_i, D_i t_i)\, f_r(t_i + \tau_{ji})\, dt_i. \quad (45)$$

We must now compute the expected cost when inventory i does not reach M_i before the cutoff time but when shortages occur after the cutoff time. Let u represent the on-hand inventory in i at time T_{ci}, the cutoff time. By assumption, $u > M_i$. For a given u, the expected number of shortages before the arrival of the first increment of the next allocation is

$$\sum_{\ell=u+1}^{\infty} (\ell - u)\, p(\ell, D_i \tau_{ji}) \quad (46)$$

The unit years of shortage can be expressed as

$$\int_0^{\tau_{ji}} \sum_{\ell=u+1}^{\infty} (\ell - u)\, p(\ell, D_i t)\, dt. \qquad (47)$$

Letting π_i and $\hat{\pi}_i$ represent the cost per backorder and per unit year of shortage in inventory i, we have the expected cost for a given u, $W_i(u)$, equal to π_i times expression 46 plus $\hat{\pi}_i$ times expression 47. Therefore, the expected cost arising when M_i is not reached before the cutoff time but when shortages occur after the cutoff time is

$$\Gamma_i^{(2)}\ (u_i - x_{ij},\ t_i) = \int_0^{\infty} \sum_{u=M_i+1}^{y_i-x_{ij}} W_i(u)\ P(u,\ D_i\tau_{ji})\ \ldots \qquad (48)$$

$$\ldots \times p(y_i - x_{ij} - u,\ D_i\ t_i)\ f_r(t_i + \tau_{ji})\ dt_i .$$

Thus the expected cost of inventory i for the remainder of the period due to shortages or to its reaching its M level is

$$U_i(y_i - x_{ij},\ t_i) = \Gamma_i^{(1)} + \Gamma_i^{(2)} \qquad (49)$$

In this problem we are concerned with only two inventories. Therefore, we will assume that once a redistribution is made from i to j during a period, there will never be a redistribution from j to i during the same period. Consequently, $R_{ji} = 0$ which will, in turn, make $\Gamma_i^{(1)} = 0$. Therefore, equation 49 becomes

$$U_i(y_i - x_{ij}, t_i) = \Gamma_i^{(2)} (y_i - x_{ij}, t_i). \tag{50}$$

We now return to inventory j to compute the expected costs pertinent to that inventory. A redistribution from i to j that is made at time t will not reach inventory j until $t + \tau_{ij}$. Since nothing can be done about shortages that occur in j during the τ_{ij} before the arrival of the redistribution, these costs need not be considered. If M_j is the on-hand inventory at j at the time a redistribution is considered, the on-hand inventory at j immediately after a redistribution arrives is

$$\theta = M_j + x_{ij} - w \tag{51}$$

where w is the demand against j in the redistribution lead time. The value of θ will depend on x_{ij} and w and can be in one of two ranges:

(a) $\theta \geqq M_j$, or (b) $\theta < M_j$.

In the first case, $\theta \geqq M_j$, terms 45 and 48 also apply, except that $y_i - x_{ij}$ is replaced by θ and t_i is replaced by $t_j - \tau_{ij}$. Hence for a given w, the expected cost is the equivalent of equation 49, i.e., $U_j(\theta, t_j - \tau_{ij})$. If we average over w we get the following expected cost at j arising from case (a):

$$\Psi_a = \sum_{w=0}^{x_{ij}} p(w, D_j\tau_{ij})\, U_j(M_j + x_{ij} - w, t_j - \tau_{ij}). \qquad (52)$$

For the case when $\theta < M_j$ recall that the rule was made that no additional redistributions will be considered in the period. Then for a given t_j, the expected shortage costs of $\theta \geqq 0$ will be of the form $W_j(\theta)$ except that the time is not τ_{ji} but $t_j + \tau_{ij} = t_a$. It is anticipated that the effect of the entire cost term for $\theta < M_j$ will be comparitively small and that the portion of the term for $\theta < 0$ will be negligible. Therefore, the computations for $\theta < 0$ will be omitted. However, the contribution for $0 \leqq \theta < M_j$ is

$$\Psi_b = \int_0^\infty \sum_{w=x_{ij}+1}^{M_j+x_{ij}} W_j(M_j + x_{ij} - w)\, p(w, D_j\tau_{ij})\, f_r(t_j + \tau_{ij})\, dt_j. \quad (53)$$

The expected costs for inventory j can now be written

$$V(x_{ij} + M_j) = \Psi_a + \Psi_b . \tag{54}$$

All the cost terms have now been evaluated. The expected costs of redistribution plus costs of reaching the M levels and the cost of shortages later in the period are

$$R(x_{ij}) = C_{ij}x_{ij} + U_i(y_i = x_{ij}, t_i) + V(x_{ij} + M_j) \tag{55}$$

for i, j = 1, 2 and $i \neq j$. This function must be minimized to find the optimal redistribution rule, i.e., the values of x_{ij} (i, j = 1, 2 and $i \neq j$).

Development of the Allocation Rule Model

A quantity Q is ordered each time the system inventory position falls to a value k. Since we have two inventories, it is necessary to allocate a portion of Q to each. Since it is desirable to delay the allocation decision as long as possible, we will assume that the allocation decision for our problem is made at the time the first increment of Q is ready to be shipped to the two inventories. Let Q_1 be the amount allocated to inventory 1 and Q_2 be the amount allocated to inventory 2 such that $Q_1 + Q_2 = Q$.

The optimal allocation will be the one that minimizes the costs of transportation from the source plus the expected costs of reaching the M levels or of shortages in the period until the next allocation. The transportation costs from the source are assumed to be constant amounts per unit shipped, i.e., C_{S1} dollars per unit shipped to inventory 1 and C_{S2} per unit shipped to inventory 2. Therefore the transportation costs can be written

$$C_{S1}Q_1 + C_{S2}Q_2 . \tag{56}$$

Suppose that y_j is the inventory on hand at j when the allocation is to be made (if there is a backlog, y_j will be negative and the number of backorders will be $- y_j$). It will be assumed that no redistributions will take place during the time allocations are being received from the factory. It has also been assumed earlier that maximum on hand inventory during any period occurs at the time of receipt of the last increment of an allocation from the factory, i e., T_{mj}. The on hand inventory in j at time T_{mj} is the $g_j = y_j + Q_j - z_j$ where z_j is the demand in time T_{mj}. The expected costs of reaching the M levels and of shortages vary, depending on whether (a) $g_j \geqq M_j$ or (b) $0 \leqq g_j < M_j$. It will be assumed that cases where there are still backlogs after completion of allocations can be ignored.

Let us first consider case (a) where $g_j \geqq M_j$. For a given z_j the probability that M_j is reached before T_{cj}, the cutoff time, is the probability that $y_j + Q_j - z_j - M_j$ units are demanded in a given time $T_{cj} - T_{mj}$, where T_{cj} is assumed to be always greater than T_{mj}. Then the expected cost of reaching M_j, after averaging over all values of z_j and T_{cj}, is

$$Y_1^{(1)}(g_i) = R_{ij} \sum_{z_j=0}^{y_j+Q_j-M_j} p(z_j, D_jT_{mj}) \; . \; . \; .$$

$$\ldots \int_0^{\infty} P[y_i + Q_j - z_j - M_j, D_j(T_{cj} - T_{mj})] \, f(T_{cj} + \tau_{cj}) \, dT_{cj}. \quad (57)$$

If we let $W_j(u)$ represent the same as in expression 48, the expected cost of shortages later in the period is

$$y_1^{(2)}(g_j) = \sum_{z_j=0}^{y_j+Q_j-M_j} P[z_j, D_jT_{mj}) \int_0^{\infty} \sum_{u=M_j+1}^{g_j} W_j(u) \, p(u, D_j\tau_{ij}) \ldots \quad (58)$$

$$\ldots \quad P[y_i + Q_j - z_j - M_j, D_j(T_{cj} - T_{mj})] \, f(T_{cj} + \tau_{ij}) \, dT_{cj}.$$

Adding expressions 57 and 58 we find that the expected cost of reaching the M level and of shortages for case (a) is

$$Y_1(g_j) = Y_1^{(1)}(g_j) + Y_1^{(2)}(g_j). \tag{59}$$

For case (b) where $0 \leqq g_j < M_j$, the expected cost can be expressed as

$$Y_2(g_j) = \sum_{z_j=0}^{y_j+Q_j} p(z_j, D_j T_{mj}) \int_o^\infty W_j(y_j + Q_j - z_j)\ f(T - T_{mj})\ d(T - T_{mj}). \tag{60}$$

The total cost to be minimized is then

$$H(g_1, g_2) = C_{S1}Q_1 + C_{S2}Q_2 + \sum_{j=1}^{2} y_1(g_j) + \sum_{j=1}^{2} Y_2(g_j) \tag{61}$$

where the $Q_j \geqq 0$ must also satisfy

$$Q_1 + Q_2 = Q\,. \tag{62}$$

Function 61 must be minimized in the presence of constraint 62 to obtain the optimal allocation rule; i.e., how much to be sent to each one of the inventories.

Development of the Warehouse Constraint Function

The final step in the development of a system model with stochastic demands is the formulation of a function to represent the constraint imposed on the system by the limited amount of warehouse space available. As in the deterministic model, the maximum on hand inventory for the system must be no larger than the warehouse space, W. If we again let w_1 and w_2 represent the amount of warehouse space per unit consumed by products 1 and 2 respectively, then for a given z_1 and a given z_2 we have

$$w_1 g_1 + w_2 g_2 \leqq W \tag{63}$$

where $g_j (j = 1, 2)$ is the maximum on hand inventory achieved upon the receipt of the last increment of an allocation at inventory j. If we average over the possible values of z_j, the warehouse constraint is

$$\sum_{j=1}^{2} \sum_{z=0}^{y_j+Q_j} p(z_j, D_j T_{mj})\, w_j(y_j + Q_j - z_j) \leqq W \tag{64}$$

Our model for the system with stochastic demands is now complete. The complete model includes the total cost function (equation

42), the redistribution rule function (equation 55), the allocation rule function (equations 61 and 62), and the constraint function (equation 64).

A Simplified Model for Determining Q and k

In the general case of the total cost function, equation 42, determining the optimal Q and k would be quite difficult. Numerical or Monte Carlo simulation procedures would be necessary. In an effort to simplify equation 42 to make a solution more easily obtained we will make some simplifying assumptions to reduce the complexity of the model. These assumptions will be similar to those made by Hadley and Whitin in the simplification of their model (12).

The first assumption that we will make is that instead of allocations arriving at the warehouse in increments over a period of time the entire lot Q arrives simultaneously. In other words we are no longer considering a finite production rate as a factor. This will cause the T_{mj}'s to be zero since the maximum on hand inventory, g_j, will occur immediately upon delivery of an allocation.

Let S_j be the safety stock for inventory j. Since the average length of a period is Q/D years and the average demand at j for the period is D_jQ/D, the average amount allocated to inventory j from each procurement is D_jQ/D. Therefore the mean of the distribution $p(g_j)$ is $S_j + D_jQ/D$. If the average lead time demand is D_T then it follows that

$$k = D_T + \sum_{j=1}^{2} S_j \tag{65}$$

We shall now use the average amount on hand immediately after an allocation, rather than the entire distribution, $p(g_j)$. We will also make the simplifying assumption that $\hat{\pi}$ and the $\hat{\pi}_j$'s are equal to zero in the terms involving system and inventory shortages and we will ignore the contributions of the expected backlog terms to the inventory carrying costs. With these assumptions equations 35 and 40 become respectively,

$$Z_j^{(1)} = \frac{R_{ij}D}{Q} \int_0^{\infty} P(S_j + \frac{D_jQ}{D} - M_j,\ DT_{cj})\ f(T_{cj} + \tau_{ij})\ dT_{cj} \quad (66)$$

and

$$Z_j^{(2)} = \pi_j D_j \sum_{x=1}^{\infty} p(S_j + \frac{D_jQ}{D} - M_j + x,\ D_jT_{cj})P(M_j - x + 1),\ D_j\tau_{ij})..(67)$$

$$\ldots \quad \int_0^{\infty} P(S_j + \frac{D_jQ}{D} - M_j,\ D_jT_{cj})\ f(T_{cj} + \tau_{ij})\ dT_{cj}.$$

In addition, the following terms in K (equation 29) go to zero:

$$\frac{1}{Q} \sum_{j=k+1}^{k+Q} \sum_{x=j+1}^{\infty} (x - j)\ p(x,\ D\tau) = 0 \quad (68)$$

and

$$\frac{\hat{\pi}}{Q} \sum_{j=k+1}^{k+Q} \sum_{x=j+1}^{\infty} (x - j)\, p(x, D_T) = 0 \,. \tag{69}$$

It can be seen that the total cost function will still be quite complex. An additional approximation will be made to further simplify TC. Instead of averaging over T_{cj}, use the mean value of T_{cj}, i.e., $(Q/D) - \tau_{ij}$. Thus if

$$\overline{X}_{cj} = D_j \, [(Q/D) - \tau_{ij}] \tag{70}$$

then equations 66 and 67 become respectively,

$$Z_j^{(1)} = \frac{R_{ij}D}{Q} \, P(S_j + \frac{D_j Q}{D} - M_j, \, \overline{X}_{cj}) \tag{71}$$

and

$$Z_j^{(2)} = \pi_j D_j \sum_{x=1}^{\infty} p(S_j + \frac{D_j Q}{D} - M_j + x, \overline{X}_{cj}) \ . \ . \ . \tag{72}$$

$$. \ . \ . \quad P(M_j - x + 1, D_j \tau_{ij}) \ P(S_j + \frac{D_j Q}{D} - M_j, \overline{X}_{cj}).$$

If we make the assumption that $g_j \geqq 0$, i.e., the on hand inventory immediately after receiving an allocation cannot be less than zero, then the range of x in equation 72 is $(1, 2, \ . \ . \ ., M_j)$. With all the above assumptions the expected yearly total cost function becomes,

$$TC = \frac{DA}{Q} + IC\left(\frac{Q}{2} + \frac{1}{2} + D\tau + \sum_{j=1}^{2} S_j\right) + \pi \frac{D}{Q}\left[g\left(D\tau + \sum_{j=1}^{2} S_j\right)\right. \tag{73}$$

$$\left. - g(Q + D\tau + \sum_{j=1}^{2} S_j)\right] + \sum_{j=1}^{2} P(S_j + \frac{D_j Q}{D} - M_j, \overline{X}_{cj}) \ . \ . \ .$$

$$\ldots \quad \left[\frac{R_{ij} D}{Q} + \pi_j D_j \sum_{x=1}^{M_j} p(S_j + \frac{D_j Q}{D} - M_j + x, \overline{X}_{cj}) \ P(M_j - x + 1, D_j \tau_{ij})\right]$$

The constraint function (equation 64) will also change as a result of the assumptions we have made. Since the maximum on hand inventory is now immediately after the receipt of an allocation, z_j is no longer a factor and $T_{mj} = 0$. Since the mean of the distribution is $S_j + D_j Q/D$, the warehouse constraint becomes

$$\sum_{j=1}^{2} w_j \left(S_j + \frac{D_j Q}{D}\right) \leqq W \ . \tag{74}$$

If equations 73 and 74 are solved for the optimal values of Q and the S_j's, then the optimal k can be found by substituting into equation 65.

CHAPTER IV

MODEL OPTIMIZATION

Having completed the formulation of the various models, the task now remains to determine the optimal operating policy for the system. For our system the optimal operating policy will entail the minimization of the various cost functions that have been formulated in order to obtain values for the following quantities:

Q - The system reorder quantity.

k - The system reorder point.

Q_1 - The amount of Q allocated to inventory 1.

Q_2 - The amount of Q allocated to inventory 2.

x_{ij} - The amount to be redistributed from inventory i to inventory j $(i, j = 1, 2 \text{ and } i \neq j)$.

In actual practice few inventory models can be efficiently optimized using purely analytical techniques. Such is the case with our model. The computer, because of the speed with which it can perform complex mathematical operations, normally plays an important role in the optimization of all but the simplest of inventory models.

The study of our system began by the development of a model for deterministic demands against the inventories. Then a model for stochastic demands was developed. This model included a complex total cost function and two separate cost functions for the redistribution and allocation rules. It was then assumed that finding the optimal

Q and k using the total cost function for the stochastic model could be quite difficult (if not impossible) and costly. Therefore, some simplifying approximations and assumptions were made. In our discussion of the optimization procedures we will first consider the deterministic model and second the stochastic model using the simplified version of the total cost function. Actual demonstration of the optimization procedures for each model will not be undertaken, although each procedure will be outlined in general.

Discussion of Deterministic Model Optimization

To determine the optimal operating policy for the deterministic system, we must find the values for Q_1, Q_2, s_1, and s_2 that minimize equation 13 (the total cost function) subject to constraint 14 (the warehouse constraint function). For this model the Lagrange Multiplier method would be appropriate. If we let ρ represent the Lagrange multiplier, then the Lagrangian function will be

$$L = K + V + B_1 + B_2 + \rho\left\{w_1\left[Q_1\left(1 - \frac{D_1}{P_1}\right) - s_1\right] + w_2\left[Q_2\left(1 - \frac{Q_2}{P_2}\right) - s_2\right] - W\right\} \quad (75)$$

where K, V, B_1 and B_2 are equations 7, 8, 11, and 12 respectively. If we now take partial derivatives of L with respect to each of the five variables (Q_1, Q_2, s_1, s_2, and ρ) and equate them to zero, we will have five equations in five unknowns. The simultaneous solution to these five equations satisfy the necessary conditions for an optimal policy.

These simultaneous equations are nonlinear. It will therefore be necessary to obtain a numerical solution by use of the computer. Most computer systems have software routines capable of solving a set of nonlinear simultaneous equations. Two well-known routines are the Newton-Raphson method and the Parameter Perturbation method (19). Both of these methods determine the roots of a set of equations by using a functional iteration process that starts from an estimated solution. The Newton-Raphson process is generally faster because of a quadratic convergence factor as opposed to linear convergence on the solution by Parameter Perturbation. However, for the Newton-Raphson method initial estimated roots must be sufficiently close to the true roots if convergence is to be obtained. This restriction does not exist with Parameter Perturbation method.

Having found the roots to the simultaneous equations (i.e., the values for Q_1, Q_2, s_1, s_2, and ρ), the optimal system reorder quantity is $Q^* = Q_1 + Q_2$, and the optimal number of backorders to incur for the system is $s^* = s_1 + s_2$. The optimal allocation rule is Q_1 units to inventory 1 and Q_2 units to inventory 2. The optimal distribution of backorders is s_1 in inventory 1 and s_2 in inventory 2. If we assume that production lead time, τ, is less than $T_3 + T_4$ (see Figure 2), then the optimal reorder point is

$$k^* = D\tau - s^* \tag{76}$$

We have now determined the optimal policy (for a given set of parameters) for a system with deterministic demands. In most cases we would be interested in observing the sensitivity of the model to changes in the parameters of the system. This can be accomplished very easily by assigning different values to the parameters of the model and repeating the same root-finding process described above.

An alternative method to solving the deterministic model is to use some direct search procedure such as pattern search, multivariate grid search, parallel tangents, etc. These methods might prove simpler to employ.

Discussion of Stochastic Model Optimization

The model for the stochastic process consists of four parts: (a) the simplified total cost function, (b) the warehouse constraint inequality, (c) the redistribution function, and (d) the allocation function. The optimal operating policy can be found through a process consisting of three phases:

(1) Determine the values of Q^*, S_1^*, and S_2^* that minimize TC (equation 73) subject to the warehouse constraint (inequality 74). Using these results compute k^*, the optimal reorder point.

(2) Determine the value of x_{ij}^* (i, $j = 1, 2$ and $i \neq j$) that minimizes $R(x_{ij})$ (equation 55).

(3) Determine the values of Q_1^* and Q_2^* that minimize $H(g_1, g_2)$ (equation 61) subject to the constraint (equation 62).

Determination of Q^* and k^*

In spite of the significant reduction in complexity of the simplified total cost function from the original (equation 42), it is still sufficiently complex to prohibit the use of an analytical approach similar to that used for the deterministic model. However, since we are interested in determining the values for only three variables (Q^*, S_1^*, and S_2^*), a near-optimal solution may be found through the use of search techniques.

One approach would be to evaluate the constrained function, TC, over a range of values for the variables (Q, S_1, and S_2). That is, for each combination of Q, S_1, and S_2, compute a value for TC. Since many of the parameters are random variables, it will be necessary to replace them by their expectations. The values of Q, S_1, and S_2 that result in a minimum TC will be considered the optimum values.

Again, model sensitivity can be tested by changing the parameter input to the program and making additional runs. In using the above approach to finding Q^* and k^*, caution must be used in the selection of ranges for the variables. For certain systems, if the ranges are made too narrow, we run the risk of finding a local minimum for TC instead of the global minimum. This problem would apply especially to systems dealing with large numbers of products where Q could conceivably take on a very wide range of values. In such a situation we would initially start with very wide ranges for the variables and use large incremental step sizes for the variables within the selected ranges to avoid over-using the computer. This would give us the general area of the global minimum. The next step would be to examine the function around the general location by reducing the step sizes and examining only the ranges of values for the variables that are in the general area of the global minimum.

To demonstrate the nature of the total cost function and to make the decision variables more identifiable, it might be helpful to assign values to the parameters. Therefore, let the parameters have the following values:

D = 10,000 units per year

A = 400 dollars per order

I = .5

C = .50 dollars per unit

τ = .005 years

π = 4.00 dollars per backorder

M_1 = 100 units

M_2 = 50 units

D_1 = 7000 units per year

D_2 = 3000 units per year

τ_1 = .003 years

τ_2 = .001 years

π_1 = 3.00 dollars per backorder

π_2 = 4.00 dollars per backorder

R_{12} = .10 dollars per unit

R_{21} = .05 dollars per unit

Substituting the above values into TC (equation 73), we obtain the following function:

$$TC = \frac{4 \times 10^6}{Q} + .25\left(\frac{Q}{2} + 50.5 + S_1 + S_2\right) + \frac{4 \times 10^4}{Q}\Big[50\ P(50 + S_1 + S_2,\ 50)$$

$$- (50 + S_1 + S_2)\ P(51 + S_1 + S_2, 50) - 50\ P(50 + Q + S_1 + S_2,\ 50)$$

$$+ (50 + Q + S_1 + S_2)\ P(51 + Q + S_1 + S_2,\ 50)$$

$$+ P\left[S_1 + .7Q - 100,\ 7000\left(\frac{Q}{10{,}000} - .001\right)\right]\left\{\frac{500}{Q}\right.$$

$$\left. + 21{,}000 \sum_{x=1}^{100} p\left[S_1 + 7Q + x - 100,\ 7000\left(\frac{Q}{10{,}000} - .001\right)\right] P(101 - x,\ 7)\right\}$$

$$+ P\left[S_2 + .3Q - 50,\ 3000\left(\frac{Q}{10{,}000} - .003\right)\right]\left\{\frac{1000}{Q}\right.$$

$$\left. + 12{,}000 \sum_{x=1}^{50} p\left[S_2 + .3Q + x - 50,\ 3000\left(\frac{Q}{10{,}000} - .003\right)\right] P(51 - x,\ 9)\right\}.$$

It can be seen that TC is still quite complex, and little insight is gained into the behavior of the function.

Determination of x^*_{ij}

Having found a value for Q^*, we can now proceed with evaluation of x^*_{ij}, the optimal quantity to redistribute from inventory i to inventory j where i, j = 1, 2 and $i \neq j$. Note that since Q is one of the parameters for $R(x_{ij})$, its optimal value must be determined first. The question of whether or not a redistribution is to be made is not considered until one of the two inventories reaches its M level prior to the cutoff time. At that time the computation is made to determine the value of x_{ij} that minimizes $R(x_{ij})$. It should be noted here that x_{ij} is a single variable; i.e., if inventory 1 has reached its M level, then $x_{ij} = x_{21}$ (i = 2 and j = 1); but if, instead, inventory 2 has reached its M level, then $x_{ij} = x_{12}$ (i = 1 and j = 2). Thus, $R(x_{ij})$ has only one variable, namely x_{ij}.

Since $R(x_{ij})$ is an even more complex function than TC, we must again resort to a numerical solution approach using computer search techniques. However, the process here is simpler because $R(x_{ij})$ is unconstrained and has only one variable (x_{ij}) to contend with. The basic approach is the same as that used for finding Q^*. Evaluate $R(x_{ij})$ for a given set of parameters and over the desired range of x_{ij}'s. Again, since many of the parameters are random variables, it will be necessary to replace them with their expected values. The value of x_{ij} that results in a minimum $R(x_{ij})$ will be considered the optimum value.

Sensitivity analysis of $R(x_{ij})$ must be coordinated with that for TC since the parameter values for both must be the same. However, the same basic procedure will be used as for TC. Again, caution must be exercised in the selection of ranges for x_{ij} because of the possible local versus global minimum dilemma.

Determination of Q_1^* and Q_2^*

The final step in determining the optimal operating policy is the allocation decision, i.e., the amount of Q to allocate to inventory 1 (Q_1) and the amount of Q to allocate to inventory 2 ($Q - Q_1 = Q_2$). The decision must be made for each cycle since Q_1 and Q_2 will vary due to the random nature of demands against each inventory and due to the redistribution decision for the previous cycle. Again, since Q is one of the parameters of $H(g_1, g_2)$, its optimal value must be computed first. Also, Q_1 and Q_2 must satisfy the constraint, $Q_1 + Q_2 = Q$.

With the above qualifications the solution procedure is exactly the same as that used for finding Q^*, S_1^*, and S_2^*. The values of Q_1 and Q_2 that result in a minimum $H(g_1, g_2)$ and that satisfy the constraint will be considered the optimum values.

Sensitivity analysis of $H(g_1, g_2)$ must be coordinated with that for TC and $R(x_{ij})$ because of common parameter values. The same caution concerning local versus global minimums should be observed.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

As a result of the research and analysis conducted in the preparation of this study, the following conclusions have been arrived at:

1. The multi-warehouse approach to formulating a model is appropriate for use in single warehouse, multi-product systems where products are interchangeable.

2. In view of the assumptions made for the deterministic model, redistribution between inventories can be disregarded as a factor in determining the optimal policy.

3. The system models are of sufficient complexity to make the optimization process one of considerable difficulty.

a. The Lagrange Multiplier method can be used for the deterministic model; however, the set of simultaneous equations formed by setting the partial derivatives equal to zero must be solved using digital computer techniques. As an alternative, direct search techniques may be employed to obtain a near-optimal policy.

b. An optimal policy cannot be found analytically for the stochastic model, but a near-optimal policy can be established using search techniques.

4. Finding an optimal operating policy for the stochastic model is greatly simplified by one's ability to decompose the model

into three sub-models that can be dealt with separately: (a) the total cost function and warehouse constraint, (b) the redistribution rule function, and (c) the allocation rule function.

5. The inventory system for which we have developed a model has several conceivable applications to inventory situations that actually exist in industry.

6. The models developed in this study substantiate the contention that models for comparatively simple inventory systems can be of considerable complexity.

Recommendations

As a result of information gained from this study, it is recommended that the following associated areas be considered as possible areas for future investigation:

1. Conduct optimization experiments for the models that have been formulated in this study. Compare the efficiency of the simultaneous equation approach, grid search, and other direct search methods such as pattern search and parallel tangents. Select the approach that offers the best overall results with respect to speed, accuracy, and ease of computation.

2. For a common set of parameters compare the optimal policy obtained for the deterministic model with optimal policy obtained for the stochastic model. Since the deterministic model is much simpler to work with than the stochastic model, it would be preferable to use it to represent the system rather than the stochastic model. If the optimal policy found using the deterministic model is reasonably close

to the policy found using the stochastic model, then the use of the deterministic model to represent the system is justified.

3. Perform sensitivity analysis on each model to determine how changes in the parameters affect the optimal policies. It is possible that changes in some of the parameters of the system can result in very significant reductions in costs. The process of altering the parameters would involve only the changing of the data input to the program.

4. Develop a model which assumes that the quantity demanded at each demand is a random variable. This would be a refinement to make the model even more representative of the true system since, in reality, the quantity of each demand is often more than one.

5. Develop a model for a system in which periodic review of inventory levels is made and an order-up-to-R reorder policy is used. Compare this model with the lot-size, reorder-point model developed in this study. It may be found that the order-up-to-R model is simpler to formulate while still providing an operating doctrine that compares favorably with the lot-size, reorder-point policy.

6. Develop a model using heuristic techniques. Compare the results obtained using this model with those obtained form the mathematical models. If the simpler heuristic model produces sufficiently accurate results, use it to represent the system.

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